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Dynamic Adjustment of Fiscal Policy under a Debt Crisis

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Abstract

In an overlapping generations framework that allows for the presence of a debt crisis scenario (debt bubbles), we introduce productive government expenditures, and endogenous deficits through a dynamic fiscal rule that combines fiscal stimulus and fiscal consolidation. We formally argue that a fiscal rule must be pro-cyclical to output for government investment financing and simultaneously has to control for the level of debt adjusting taxation for a policy aiming to escape a situation of exploding debt and low economic activity. Then, when the economy becomes sustainable (or in an economy's high initial private capital), the same rule, has to endogenously adapt to the actual level of debt and income in order to stimulate private investment through lower taxes. We provide a numerical example for our theoretical results and show that in economies with sufficiently low levels of capital and high levels of debt, the tax rate has to adjust non-monotonically during the recover process, reflecting the two counter-balancing properties of the examined fiscal policy rule. However, under a threshold level of initial capital stock, taxes must adjust monotonically (negatively) to boost private investment activity, and in turn, alleviating the volume of debt though a higher tax base.

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1 Introduction

In recent years, particularly in Europe, we witnessed a shift to austerity measures and deficit reducing policies to target sustainable public debt. The International Monetary Fund (IMF), the European Central Bank (ECB), and the European Commission, in an effort to help European countries overcome exploding debt, focused on policies that place some level of fiscal austerity (increase in taxation and spending cuts) to control each country's debt. However, we first saw that those policies, due to their discretionary nature, are continuously re-optimized given some countries' failure to achieve their targets. Second, using almost the same fiscal policy measures in similar countries (e.g., Portugal and Greece), we observe diverging economic outcomes (for a detailed review, see Brendon and Corsetti, 2016). This variation in the dynamic adjustment of policy instruments and divergence from the expected economic outcomes resulted in an uncertain economic environment, raising the need to impose a stable dynamic fiscal policy rule subject to the state of the economy.

This study aims to examine the properties of a fiscal policy rule for debt sustainability in a framework that allows for the presence of debt bubbles. On the one hand, increasing productive government spending stimulates an economy with low private investment and, in turn, output. On the other hand, without considering a consistent financial plan for the level of debt, an expansionary policy can generate a debt bubble. According to our model, the effectiveness of fiscal stimulus and consolidation for debt sustainability is determined by the initial conditions of the level of debt and capital stock. To this end, we provide a fiscal rule that can endogenously adjust to the need for stimulus and consolidation as the economy develops. In particular, we show that a fiscal rule has to be pro-cyclical in output increases (contrary to perceived notions) through public investment (”productive” stimulus). However, at a high initial level of debt, taxation has to increase (endogenously) in order to finance deficits.[1] If the economy achieves (or starts) a threshold (or initial) level of capital stock, taxation negatively adjusts to output increases and government expenditures are financed through a higher tax base.

Our study is related to the literature on fiscal consolidation and debt sustainability and contributes in several respects. Earlier work by Sargent and Wallace (1981) states that there is a ceiling on government indebtedness and that permanent deficits will eventually need to be monetized. However, some countries either belong to a monetary union or monetary policy is constrained by the zero lower bound. Thus, Eggertsson (2011); Christiano, Eichenbaum, and Rebello (2011); and Coenen et al. (2012), among others, highlight the role of fiscal stimulus and show that government spending multipliers are potentially larger when the zero bound is binding. However, their modelling approach does not allow for the presence of debt bubbles that fiscal stimulus can trigger and the fact that fiscal multipliers depend on the state of the cycle (Ramey and Zubairy, 2016) and the level of debt according to empirical evidence provided by Auerbach and Gorodnichenko (2012, 2013) and Fotiou (2017). Furthermore, Corsetti et al. (2013) highlight that the benefits to fiscal expansion could easily be undone if the fiscal solvency of the government comes into question – an issue of obvious relevance to Southern European countries at present.

[1] In the optimal neoclassical growth model of infinitely lived agents debt bubbles are ruled out optimally and a procyclical fiscal rule crowds out private investment strongly and generates instability. However, under the existence of debt bubbles and unstable debt dynamics that can occur in an OLG framework, a procyclical policy in output can place the economy in the sustainability area (through increases in productivity) as we will show later on.

To this end, driven by the aforementioned empirical evidence, we first allow for the presence of debt bubbles (following Tirole, 1985 and Chalk, 2000) to account for the unfavorable consequences of fiscal stimulus on debt. Second, through a policy rule, we consider state dependent fiscal stimulus (alternatively, through productive government expenditures) to remedy a recession while also controlling for the level of debt. We contribute a theoretical framework that formalizes the effectiveness of fiscal policy under the state of cycle and debt. Our new mechanism puts forward an analysis of the differential dynamic paths conditional to the initial conditions.[2]

In particular, regarding the literature on fiscal sustainability in a framework that allows for self-fulfilling bubbles, the closest work to ours is that of Chalk (2000). Once debt can be rolled over to generations (the Ricardian equivalence does not hold), Chalk (2000) studies the maximum level of permanent deficits (empirically observable in the US and elsewhere) that a country can run subject to its structural characteristics and initial conditions. Interestingly, the author shows that the level of permanent deficit that a country can afford depends on its inherited level of debt. However, Chalk (2000) assumes that deficits are exogenous and constant and thus ignores the implementation of deficits over time. We complement this study by endogenizing deficits through a policy rule, estimated empirically by Bohn (1998), that considers the source of deficits through the positive response of deficit to output (in our case, to finance productive expenditures). Second, the rule activates a positive response in taxation to increases in debt to control the emergence of a debt bubble.[3] To this end, we can investigate when and what determines the choice of stimulus and consolidation to bring the economy into sustainability. Furthermore, we examine the effect of those counter-balancing properties of the examined fiscal policy on the dynamics of taxation.

Regarding the literature on fiscal stimulus and consolidation above, our framework highlights the importance of productive government spending (in the spirit of a Samuelson (1959) rule) and its interaction with the state of output (capital) and debt.[4] Surprisingly, although past empirical studies show that government spending has a positive effect on the productivity of capital (Aschauer, 1989; Nadiri and Mamuneas, 1994; Lepper et al., 2010), recent studies assume that government expenditures are entirely wasteful and have no direct effect on the marginal productivity of private inputs (Eggertsson, 2011; Christiano, Eichenbaum, and Rebello, 2011; Coenen et al., 2012). Recent exceptions are works by Traum and Yang (2015) and Bouakez, Guillard, and Roulleau-Pasdeloup (2017). Traum and Yang (2015) consider expansionary fiscal policy shocks and, contrary to the conventional view (and in the spirit of our paper), show that if a debt expansion is due to an increase in government investment, private investment rises within the first three years, despite a higher interest rate. Along the same line, Bouakez, Guillard, and Roulleau-Pasdeloup (2017) focus on the effectiveness of public investment in stimulating an economy stuck in a liquidity trap. They estimate a positive impact of government investment on private productivity. Additionally, they find that the spending multiplier associated with public investment can be substantially large in a crisis scenario. Both frameworks exclude the possibility of a debt bubble, which

[2]Main computable macroeconomic models on the topic consider a stable and unique path for any initial condition (among others, Eggertsson, 2011, Christiano, Eichenbaum and Rebello, 2011 and Coenen et al., 2012).

[3]This second feature of the rule provides an additional channel to deficit control when structural deficits, in the spirit of Chalk (2000), are close to achieve their upper bounds.

[4]Interestingly, the largest fiscal stimulus plan in U.S. history — the American Recovery and Reinvestment Act (ARRA) of 2009 — allocated roughly 40% of non-transfer spending to public investment.

generates differential self-fulfilling dynamics for initial conditions far from the steady-state. To this end, complementing the evidence and results from these studies, we advance the role of government investment and its association with (productive) stimulus under a threat of a possible debt bubble. The interaction of initial conditions (state of cycle and debt) with the (endogenous) efficiency of productive spending expands the set of mechanisms in the design of a fiscal plan that aims for debt sustainability.

Regarding policy implications, we argue that fiscal asymmetries may not rely solely on fundamentals but on self-fulfilling pessimism derived from initial conditions as recent empirical evidence indicates (De Grauwe and Yuemei, 2013). In particular, we show that multiple equilibria (Azariadis and Stachurski, 2005) can arise, and although countries have similar characteristics (e.g., Spain, Italy, Portugal, and Greece) and follow similar policies, they may face divergent paths in debt and income, as observed in the data (Brendon and Corsetti, 2016; Fotiou, 2017). Our policy implications are in line with Favero and Giavazzi (2007), where the absence of a debt feedback effect on taxes and government spending can result in incorrect estimates of the dynamic effects of fiscal shocks. However, we advance their study by theoretically showing that the feedback effect of debt on taxes may not be monotonic subject to the initial conditions. Once those non-linearities (phase of business cycle and level of debt) are accounted for, empirical studies may provide more precise results regarding fiscal multipliers (see, e.g., Fotiou, 2017). Last, Balke and Ravn (2017) numerically analyze the Greek debt crisis case and find that under the existence of a commitment device from a third party in a period of crisis, the fiscal stimulus recipe is optimal relative to austerity measures from spending cuts. Our result is in line with Balke and Ravn (2017); however, our mechanism does not rely on the demand effects of fiscal stimulus but on the concept of productive reforms to the public sector.

Section 2 sets up the model and Section 3 examines the existence, uniqueness and stability of equilibria. Section 4 investigates the effect of the policy parameters on steady-state output and provides a simple numerical example about the short-run dynamics. Section 5 discusses the role of economic fundamentals and initial conditions in the determination of the parameter values of the fiscal plan and proposes research directions. Section 6 concludes the paper.

2 The Model

2.1 Demand Side

We consider an overlapping generations model as in Diamond (1965) and Tirole (1985). There are N_t consumers who each live for two periods. They choose their consumption today, C_t, and tomorrow, d_{t+1}, to maximize intertemporal utility as given by the following utility function:

$$U = \ln C_t + \beta \ln d_{t+1} \tag{1}$$

where $\beta > 0$ is the weight that agents place in their second period utility. In the first period of their lives, agents inelastically supply labor and receive a wage of, w_t, which is taxed at τ_t. When old, the agents consume their savings and receive a return on their savings, r_{t+1}. By solving their intertemporal problem, the savings, S, of each individual are positively determined by the after-tax wage rate and their savings propensity, $s \equiv \frac{\beta}{1+\beta}$,

$$S(w_t) = s(1-\tau_t)w_t \tag{2}$$

2.2 Supply Side

On the supply side, there exists a continuum of firms that produce output, y_t, using capital, k_t, labor, l_t, and a public good supplied by the government g_t,

$$y_t = Ak_t^{\alpha} l_t^{1-\alpha} g_t^{\gamma} \qquad \alpha + \gamma < 1 \tag{3}$$

where $\alpha > 0$ denotes the share of physical capital on the production function. Following empirical evidence from Aschauer (1989); Nadiri and Mamuneas (1994); and Bouakez, Guillard, and Roulleau-Pasdeloup (2017), among others, we assume that the share of public spending on production,$\gamma > 0$, is positive. The wage rate and real return on capital, R_t, using the labor market clearing condition, $l_t = 1$, are determined by

$$w_t = (1-\alpha)Ak_t^{\alpha} g_t^{\gamma} \tag{4}$$

$$R_t = \alpha A k_t^{\alpha-1} g_t^{\gamma} \tag{5}$$

2.3 Government

Following Pestieau (1974) and Turnovsky and Fisher (1995), we assume that the supply of the public good is determined by a production efficiency rule in the spirit of Samuelson (1959), which states that the marginal benefit generated by the public good (expenditures) must be equal to the marginal cost of its production, given by:

$$g_t = (\gamma A k_t^{\alpha})^{\frac{1}{1-\gamma}} \tag{6}$$

According to (6), government expenditures adjust through time subject to the level of the capital stock. In particular, the marginal benefit of public funds decreases as capital increases. Using (6), we can compute the equilibrium wage and real interest rate as follows:

$$w(k_t) = (1-\alpha)Ak_t^{\alpha} g_t^{\gamma} = (1-\alpha)\tilde{A}k_t^{\frac{a}{1-\gamma}} \tag{7}$$

$$R(k_t) = \alpha A k_t^{\alpha-1} g_t^{\gamma} = \alpha \tilde{A} k_t^{\frac{\alpha}{1-\gamma}-1}, \tag{8}$$

where $\tilde{A} \equiv \gamma^{\frac{\gamma}{1-\gamma}} A^{\frac{1}{1-\gamma}}$. Further, we assume that the government finances public expenditures not only from taxation but also by issuing government debt, B_t. The government's budget constraint is given by

$$B_{t+1} = R_t B_t + g_t - \tau_t w_t \tag{9}$$

. Following the fiscal rule estimated by Bohn (1998) and similar to Gali et al. (2007), we assume that the primary surplus/deficit is a function of the level of debt and income determined by the fiscal policy parameters, $a > 0$ and $b > 0$ given by

$$g_t - \tau_t w_t = -aB_t + by_t \tag{10}$$

. Policy parameter a measures the responsiveness of the deficits to the level of debt ("debt control" channel) and parameter b measures the responsiveness of deficit in the level of income ("fiscal stimulus" channel). Thus, this rule places some level of fiscal discipline, "austerity," as given by a, in the sense that under an increase of debt, taxation has to increase to reduce the deficit and, in turn, public debt. On the flip side, as the economy develops, policy parameter b allows for a higher structural deficit in order to finance public spending.[5]

3 Equilibrium Dynamics

Given that in equilibrium saving must be equal to investment in real capital and government bonds, after some algebra (see Appendix 1), the dynamic equilibrium is determined by the following dynamic system of equations

$$k_{t+1} - k_t = (s(1-\alpha) + s(b-\gamma) - b)\, y(k_t) - k_t + (a(1-s) - R_t(k_t))\, B_t \tag{11}$$

$$B_{t+1} - B_t = (R(k_t) - a - 1)B_t + by(k_t) \tag{12}$$

The steady-state of capital stock and debt level in the economy is the bundle, (k^*, b^*) such that $k_{t+1} - k_t = 0$ and $B_{t+1} - B_t = 0$ hold simultaneously.

Proposition 1 *(Existence and Uniqueness). For a given value of structural parameters in their assumed domain, for (i)* $b < \frac{s(1-\alpha-\gamma)}{(1-s)} \equiv b^{\max}$ *and (ii)* $a < \frac{\alpha}{(1-s)(s(1-\alpha-\gamma)-(1-s)b)} \equiv a^{\max}$ *there exist two non-trivial equilibrium steady states,* $k_{ss}^{low} > 0$ *and* $k_{ss}^{high} > 0$*, where* $k_{ss}^{low} < k_{ss}^{high}$.
Proof. *Appendix 2.* ▪

Proposition 2 *(Stability) Both steady-states are stable; the lower equilibrium,* k_{ss}^{low}*, is saddle-path stable and the higher equilibrium,* k_{ss}^{high}*, is a stable node.*
Proof. *Appendix 3.* ▪

Proposition 1 shows that there exist two non-trivial equilibria and sets up the upper bound limits of the policy rule parameters for the existence of an interior equilibrium. Condition (i) restricts the government to a certain upper limit in the use of structural deficits, $b^{\max}$, above, in which investment in capital is fully crowded-out by government spending, resulting to a non-interior equilibrium.[6] Condition 2 sets the upper limit of the response in taxation to debt accumulation to reduce deficits (limits for "austerity"). If taxation increases highly in response to debt ($a > a^{max}$), then the initial benefit from the reduction in the deficit and debt is structurally outweighed by the reduction in private savings.

Proposition 2 examines the stability of equilibria and shows that both equilibria are stable. However, the relatively lower one displays saddle-path stability and is thus a meaningless equilibrium.[7] The higher equilibrium is a stable node; thus, there exists a neighborhood of

[5]Expessing the primary deficit as a ratio of y_t we obtain $\frac{g_t - \tau_t w_t}{y_t} = -a\frac{B_t}{y_t} + b$. Thus, $b > 0$ is the part of deficit to income ratio that is structural. Interestingly, we show later on that even in the presence of a positive structural deficit (which is the case in many European countries) our rule is able to place the economy in a sustainable path.

[6]This can be seen from the first argument in the right hand side of (11)

[7]It is meaningless as it can be attained only by very certain initial conditions which cannot be chosen optimally by the agents (see among many others, Azariadis and Stachurski, 2005).

initial conditions leading toward it. According to these stability properties, Propositions 1 and 2 together imply that the initial conditions, the initial level of debt, and physical capital determine the long-run position of the economy.

Specifically, the phase diagram in Figure 1 illustrates our analytical results. For given structural parameters, and for initial conditions of k_t and B_t located on the left of the stable arm (thick dotted lines) the dynamics (represented by the arrows of motion) lead to explosive debt and low output. This happens as, in this area, for a given a and b, fiscal stimulus is weakly (sufficiently low a) financed by taxation and activates a vicious cycle of uncontrolled deficits, higher debt, and lower output (for example, see point B of Figure 1). While, at the area of initial conditions on the right of the stable arm, the stable node attracts the dynamics, as capital and output are sufficiently high to repay a sufficiently low level of debt (e.g., Point A of Figure 1). The position of capital and debt zero locus depends on the structural parameters of the economy, which are given, and the level of the policy rule parameters, which are policy variables and can alter the dynamics of a point in the phase diagram.

4 Policy Effects and Implications

In this section, we investigate how changes in the policy parameters affect the two equilibria. We show how the policy rule parameters alter the position of an economy in the phase plane (shift of the k-locus and/or b-locus in Figure 2). Then, we analyze and explain the short-run dynamics and the associated policy implications.

Proposition 3 *The policy parameter a, negatively affects the relatively lower steady state, k_{ss}^{low}, and, positively affects the higher steady-state equilibrium, k_{ss}^{high}.*
Proof. *Appendix 4 provides the proof.* ▪

Proposition 3 states that a change in policy parameter a has opposing effects on the equilibria derived in Proposition 1. An increase in the "austerity" parameter, a, decreases the relatively lower steady-state of the capital stock while it increases the relatively higher steady-state capital stock. In other words, the higher the responsiveness of the tax rate to the level of debt, the higher the gap between the two equilibria is. This implies a minimum value of a that can place the lower equilibrium away from an economy trapped in an unsustainable path.

To advance the intuition of this result, we introduce Figure 2 in comparison with Figure 1. As Figure 1 shows, an economy with a high initial level of debt and low initial level of capital stock (Point B) moves towards an unsustainable path. Close inspection of the arrows of motion shows that, initially, the level of output and the level of debt increases, and then debt explodes, negatively affecting capital. This happens because at a low initial level of capital stock, the marginal benefit of government spending is high (from equation 6) and the policy rule activates fiscal stimulus, which increases private productivity, capital, and output. However, as government spending increases, the interest rate increases (see equation, 5) and induces an increase in debt. At a high initial condition of debt, interest payments increase and debt explodes, unless taxation responds sufficiently (sufficiently low a) to finance the increased government spending (rather than through additional issue of debt). In that case, a higher level of a is required so taxation will increase sufficiently to reduce deficits and control the increase in the level of debt. This minimum level of required a depends on how far the initial level of debt and capital stock are at the left of the stable arm. In particular, following Figure 2, a sufficient increase in a shifts the k-locus (and, in turn, the stable arm) upwards and the country in point B enters the area of sustainability.

To better illustrate our analytical results, we provide a simple numerical example using standard parameter values from the growth literature. Assume two countries, A (e.g., Spain) and B (e.g., Greece). Both countries have the same structural characteristics, such as total factor productivity, $A = 8$ (scale parameter), share of physical capital in the production function, $\alpha = 0.25$ (range 0.2-0.4 in growth literature), productivity of the public good, $\gamma = 0.15$ (Bermperoglou, Pappa and Vella, 2017), time preference, $\beta = 0.99$, and both follow the same rule with weights $a = 0.5$ and $b = 0.013$ (using values close to the estimates by Bohn, 1998). Additionally, both countries are developed in the sense that they belong to the area of low interest rates (right hand side of the discontinuity in Figure 1). They only differ in their initial level of debt and capital stock. Country B has relatively lower initial capital stock, $K_0^B = 0.5$, and a higher initial level of debt, $B_0^B = 0.3$, , than country A does at $K_0^A = 3$ and $B_0^A = 0.2$. Then, our numerical exercise examines how the dynamics of each country evolve. Figure 3.1 shows that Country A reaches the high steady-state of capital stock with a sustainable steady-state level of debt, while in Figure 3.2, Country B ends up with exploding debt. Thus, Proposition 1 and 2 imply that although these countries have the same structural characteristics and follow the same policy rule, they display different dynamics (and long-run position) just by starting with different levels of inherited debt and capital stock.

[Insert Figure 3.1 about here]

[Insert Figure 3.2 about here]

Following the intuition described above, the policy implication from this result is that, for the policy rule design, the choice of the level of austerity has to depend not only on the fundamentals, but also on the initial state of the inherited debt and income. Thus, in cases of exploding debt, following Proposition 3, countries have to increase the response of taxation to the level of debt, a, so as to expand the area of sustainability, as Figure 2 shows. Figure 3.3 illustrates the dynamic path of Country B by only increasing the level of a from 0.5 to 0.9.

[Insert Figure 3.3 about here]

According to Figure 3.3 (or/and Figure 2), with higher a, the policy rule can place Country B on a stable path associated with sustainable long-run levels of debt and capital. An interesting outcome is the endogenous non-monotonic dynamics of the tax rate, which increases at low levels of capital stock, decreasing the deficit and stabilizing the level of debt. As debt falls and income increases, taxes fall in order to boost savings, which form a higher capital stock and a higher tax base to finance government expenditures. In other words, the two features of the rule work as follows. On the one hand, productive government spending increases private productivity, but it is financed through sufficiently higher taxes to control the level of debt. On the other hand, after a threshold level of capital stock, the relative return on private investment on output increases (the marginal benefit of public funds falls) and taxation decreases to boost private savings. Compared to other policy rules (which work in environments of stable and unique dynamic paths) where deficits have to decrease as output increases (for consumption smoothing), this rule provides fiscal stimulus to enable the economy to escape from low income, but guarantees some level of fiscal consolidation to prevent the emergence of debt bubbles.

Last, we do the same work for the policy instrument that controls the level of structural deficit, which Chalk (2000) analyzes extensively in a similar vein.

Proposition 4 *The structural deficit parameter, b, positively affects the lower steady state k_{ss}^{low} and negatively affects the higher, k_{ss}^{high}, steady-state equilibrium.*
Proof. *Appendix 4.* ▪

Proposition 4 states that the equilibria display different properties under a change in the response of the deficit to the level of output, b. An increase in the level of b positively affects the low steady-state and negatively affects the high steady-state. A reduction in b expands the area of sustainability. This theoretical result conforms to Chalk's (2000) result. A decline in the level of structural deficit positively affects the level of the higher capital stock because at a high level of capital stock, the productivity of capital is relatively higher than that of public investment. Thus, the lower required provision of government services (lower structural deficits) promotes tax reduction, which increase the relatively more productive private investment, raising the capital stock accumulation, and in turn, the steady-state, k_{ss}^{high}. At the same time, if deficits respond less to output, starting at a lower initial capital stock, where government spending has higher returns on output, government spending is under-financed (from the lower response of deficits to output, b) and the productivity of capital decreases, leading to a lower steady-state of k_{low}. However, note that a reduction in the level of structural deficit, b, is not always politically feasible. Under political constraints (e.g., strong unemployment benefits) governments are constrained in their use of a high level of b. In such a case, both Chalk (2000) and our condition (i) of Proposition 1 show that if b approaches its upper limit, an interior equilibrium is violated. In our model, this raises the importance of the alternative policy parameter, a (absent in Chalk, 2000), the implications of which we analyzed above.

5 Discussion and Further Research

In this section, we briefly discuss the importance of the initial conditions and other structural parameters for the level of the policy rule parameters required for debt sustainability.[8] In addition, we discuss further research directions.

5.1 The importance of the initial conditions

The main message of our study is that policymakers interested in the sustainability of various countries' debt need to look at not only its specific structural parameters but also their initial

[8]We provide a Companion Appendix with a detailed analysis of this section together with numerical examples that we discuss in this section.

conditions. Different to recent macroeconomic models that assume unique equilibria regimes and an absence of debt bubbles (they do not allow governments to roll over debt to future generations), we showed that the initial conditions play a crucial role in determining the dynamics towards sustainability.

Shedding more light on this discussion, in our model, a fiscal plan has to account for *both* of the initial conditions for the state variables in the economy, as each *alone* is *not sufficient* for sustainability. In the previous example, we considered an experiment in which countries A and B had different initial conditions for both capital and debt. However, the same difference in stability conditions can arise by varying only the initial level of capital (or debt). According to the phase diagram (Figure 1) that summarizes our analytical results, for the bundles of initial conditions on the left (right) of the stable arm, the economy follows a trajectory to unsustainability (sustainability). For example, if we fix the level of debt (i.e., the same for both countries), then, by varying only the capital stock, we can show that one country can be on the left of the stable arm (bubble area) and the other can be on the right of the stable arm (sustainability area). These economies will feature different dynamics, as in the numerical example above. Along the same line, even if two countries have the same high capital stock, but differ in their level of inherited debt (one high and one low), they will display different dynamics.[9]

Regarding the adjustment dynamics, the *level* of the policy rule parameters can place an economy that faces unsustainable dynamics in various locations on the phase plane, generating differential dynamics. In particular, in a country with moderate capital stock but a high level of debt, a moderate increase in a will result in non-monotonic adjustment dynamics of taxes following the rationale of our main example. However, under a strong increase in a, lower taxes are necessary to boost private savings through a monotonic decrease in the tax rate.[10] In other words, under a moderate level of capital stock, a strong increase in a signals strong fiscal discipline and allows debt financing through lower taxes that increase savings, capital, output, and, in turn, the tax base. To sum up, the level of austerity, a, is not only important for an economy to escape an unsustainable path, but it is also important for how (dynamics) taxation (and the other endogenous variables) adjusts towards the sustainable equilibrium.[11]

5.2 Effect of structural parameters on the determination of the fiscal plan

The structural parameters in our model play an important role in the *quantitative determination* of a and b that aim for sustainability.[12] In this subsection, we first consider the effect of the *responsiveness of private agents to taxation* on the *level* of a required for sustainability. We do this by relaxing the assumption of an inelastic labor supply. Second, we examine the implications of changes in the *responsiveness of output to government spending* on the level

[9]We present both cases (differences either on debt or on capital alone) in the Companion Appendix, Section 1, along with numerical examples for each case.

[10]See Figure A2.3.1 vs A2.3.2 in the Companion Appendix for a numerical illustration)

[11]Graphically, according to the shift in k-locus (driven by changes in a), an economy can be placed either between the intersection of the k-locus with the B-locus or outside their intersection where the dynamics according to the arrows of motion are different.

[12]The qualitative implications of our model are not affected by the values of the structural parameters in their assumed domain.

of a and the adjustment dynamics of taxation.

5.2.1 Responsiveness of Private Agents to taxation

We assume a simple framework with endogenous leisure where individuals, in addition to our main model, obtain utility from leisure, n_t, given by the following augmented utility function, $\tilde{U}$:

$$\tilde{U} = \ln C_t + \beta \ln d_{t+1} + \chi \ln n_t, \tag{13}$$

where $\chi > 0$ denotes the preference for leisure. Following the other assumptions of our main model, the savings of individuals are given by:

$$\tilde{S} = \tilde{s}(1 - \tau_t)w_t, \tag{14}$$

where $\tilde{s} \equiv \frac{\beta}{1+\beta+\chi}$ is the augmented savings propensity, which is a negative function of the preference for leisure $\frac{\partial \tilde{s}}{\partial \chi} < 0$. Solving the demand side of our model, the wage rate is given by

$$w_t = (1 - \alpha)\hat{A}\left(\frac{1+\beta+\chi}{1+\beta}\right)^{-\alpha} k_t^{\alpha}, \tag{15}$$

where $\hat{A} \equiv A(\frac{1+\beta}{1+\beta+\chi})^{1-\alpha}$. Then, in equilibrium, savings are given by

$$\tilde{S}(k_t) = (1 - \alpha)(1 - \tau_t)\hat{A}k_t^{\alpha} \tag{16}$$

where the preference for leisure negatively affects equilibrium savings, $\frac{\partial \tilde{S}}{\partial \chi} < 0$. Importantly, an increase in leisure preference lowers the response of savings to taxation $\frac{\partial \tilde{S}}{\partial \tau} = -\tilde{s}w_t = -(1-\alpha)\hat{A}k_t^{\alpha}$ (i.e. the higher χ is, the lower $\hat{A}$ and lower $\frac{\partial \tilde{S}}{\partial \tau}$ become). The dynamics of this framework are qualitatively equivalent to our main model (see Companion Appendix). However, the impact of taxation on private sector savings, which is determined by the preference for leisure, provides interesting *quantitative implications.*

In particular, the response of savings to taxation determines the level of the crowding out of private investment from an increase in taxation. We want to highlight that *the higher (lower) the* χ, the *lower the responsiveness of savings to taxation* is. The intuition is as follows. First, the higher the χ is, the lower the propensity for savings, $\tilde{s}$ ($\frac{\partial \tilde{s}}{\partial \chi} < 0$) and, in turn, the lower the responsiveness of savings to taxation, $\frac{\partial S}{\partial \tau} = \tilde{s}w_t$. Second, the higher the χ is, the lower the wage income ($w_t l_t$) is (from lower labour supply, $\frac{\partial l_t}{\partial \chi} < 0$), and, in turn, the lower the loss of savings (that are function of wages) is from higher taxation. Due to both mechanisms, when leisure preference, χ, is relatively high, the negative impact of taxation on savings declines.

The analysis above has interesting quantitative implications for the level of a targeting sustainability. For example, consider two countries that must increase a (more aggressive taxation) to enter the sustainability area. According to the analysis above, the economy that has a higher preference for leisure, χ, can afford a higher increase in a as savings are less responsive to taxation. The opposite happens for the economy with a lower preference for leisure, where the increase in a must be weaker to avoid distorting the more responsive private savings to taxation.

To sum up, the main quantitative implication of the augmented model is that *the higher (lower) the* χ is, the *lower (higher) the responsiveness of savings to taxation* is, *the higher (lower)* a *has to be for an economy to enter the area of sustainability.*[13]

5.2.2 Response of output to government spending

The main ingredients of the output response to government spending include (i) the "structural," measured by the share of government spending in the production function, γ, and (ii) the "endogenous," the differential effect of spending on output according to the initial conditions of the economy. Both channels are important for the quantitative determination of the parameters of the fiscal plan and the adjustment dynamics of taxation.

As we study above, the initial conditions of the capital stock affect, through the Samuelson rule, the relative return on output from a change in public versus private investment spending. In our model, the marginal benefit of government spending on capital stock and, in turn, output, is relatively higher at a lower level of capital stock (see 6 and Bouakez, Guillard, and Roulleau-Pasdeloup, 2017). In turn, for a low level of K, excessive debt has to be financed through a higher tax rate that controls the level of deficits rather than through spending cuts. Then, following our main numerical example, after a threshold of capital, taxation reduces to financing the relatively more productive private investment spending. Using the same logic, when a country is relatively highly endowed in capital and again high debt (still in the area of unsustainability), then, to enter the sustainability area, taxation must decrease from the beginning because private investment (endogenously) has a higher return on output and can generate a higher tax base to finance deficits. Thus, the adjustment dynamics of taxation depend on the initial conditions of capital, which endogenously determine the response of output to government spending. [14]

According to the "structural" ingredient, the lower the structural productivity of government spending, γ, is, the lower the effect of fiscal stimulus on output is. In turn, the lower the value of a for debt sustainability has to be, as the fiscal stimulus channel is weaker. In other words, taxation has to respond with a lower amount to increases in debt if the effect of fiscal stimulus through government spending is weak (lower γ).[15]

To sum up, according to the Samuelson rule, the level of capital stock has a negative effect on the marginal benefit of government spending. This endogenously determines the response of output to government spending. Since the dynamic adjustment of taxation depends on the marginal benefit of government spending relative to the crowding out effect from taxation, the initial level of capital stock (and debt) plays an important role. In particular, the adjustment dynamics become non-monotonic or monotonic according to the combination of initial conditions for the state variables. Last, because parameter γ affects the strength of fiscal stimulus structurally, it has important quantitative implications in the selection of fiscal policy rule parameters.

[13] In the Companion Appendix (Section 2), we provide analytical and numerical examples of this result.

[14] We provide numerical results of this analysis in Subsection 1.2 of the Companion Appendix (Figure A2.3.2).

[15] Numerical results are similar to the case of the leisure preference parameter and are available upon request.

5.3 Further Research Directions

Our theoretical model provides a basis for further research in the topic. An interesting research direction is the calibration of our framework for countries in the European periphery and the estimation of the policy rule parameters for debt sustainability. To this end, introducing automatic stabilizers, for example, unemployment benefits, is very relevant for the European case. Intuitively, the structural deficit parameter b captures the automatic stabilizers in our model. For instance, higher unemployment benefits have a positive impact on the value of structural deficits, b, and, according to Proposition 4, reduce the sustainability space. In such a case, our model would propose a higher level of fiscal austerity. However, as fiscal austerity is not always politically feasible, a neutral revenue reallocation of government revenues from utility enhancing to productive spending can alleviate the need for higher taxes by reducing the level of automatic stabilizers.

Furthermore, to our surprise, recent empirical evidence about the productivity of public capital is very limited (and absent for the European periphery), while our study shows that it is crucial, together with the initial conditions, in the determination of a sustainable fiscal plan.

Last, once the introduction of a policy rule raises welfare changes, the optimal selection of the fiscal rule parameters provides a promising research direction. Although the Samuelson rule seems welfare enhancing in both generations (following our numerical examples), its optimal selection in a heterogeneous agents framework will provide interesting political economy considerations.

6 Conclusion

Motivated by the recent debt crisis in the Eurozone, we provided a theoretical framework that allowed for the presence of debt bubbles. In addition, we endogenized deficits through ”productive” fiscal stimulus, and in contrast to recent discretionary policy making, we introduced a policy rule that endogenously responds to the level of stimulus and debt required for sustainability. We formally argued that the initial conditions of debt and capital are crucial in the design of a fiscal plan for debt sustainability and higher output.

In particular, under relatively high levels of debt and low levels of economic activity, while fiscal stimulus (productive government expenditures) increases the economy's production, higher taxation is required to control the level of deficits and prevent the emergence of a debt bubble. However, if the capital stock is (or becomes) relatively high, taxation must decline in order to boost private investment. Interestingly, the adjustment dynamics of taxation depend on both the structural parameters and the initial conditions, something that recent computable macroeconomic models (that consider a unique equilibrium path) do not consider at the same time. Countries with the same structural characteristics, even if they follow the same fiscal plan, may diverge due to differences in the level of inherited debt and the state of the economic cycle, consistent to what we experienced in the Eurozone.

We believe that our theoretical results bring interesting testable predictions for empirical and quantitative research.

7 Appendix

Appendix 1: Derivation of the Dynamical System

Given that in equilibrium saving must be equal to investment in real capital and government bonds, the dynamic equilibrium is given by the following dynamical system

$$B_{t+1} = R(k_t)B_t - aB_t + by_t$$

$$k_{t+1} + B_{t+1} = s(1-\tau_t)w(k_t)$$

and equation (10). Given that the government follows a Samuleson rule to determine the public spending, the marginal tax needs to adjust to implement the fiscal rule,

$$g_t - \tau_t w_t = -aB_t + by_t$$

where dividing by y_t

$$\frac{g_t}{y_t} - \tau_t \frac{w_t}{y_t} = -a\frac{B_t}{y_t} + b$$

and solving for τ_t the marginal tax is equal to

$$-\tau_t = -a\frac{B_t}{y_t} + b - \frac{g_t}{y_t}$$

. Also, from (6) and the production function we get

$$\frac{g_t}{y_t} = \gamma$$

. Then, we have that

$$-\tau(k_t, B_t)w_t = -aB_t + (b-\gamma)y_t \tag{17}$$

. We simplify the expression for k_{t+1} using eq. (17) and $\frac{w_t}{y_t} = 1 - a$,

$$k_{t+1} - k_t = (s(1-\alpha) + s(b-\gamma) - b)\, y(k_t) - k_t + (a(1-s) - R_t(k_t))\, B_t$$

which is equation (11).

Appendix 2. Existence and Uniqueness

From (11) and (12) the change of capital stock and the debt level of the economy is determined by the following dynamic system:

$$k_{t+1} - k_t = (s(1-\alpha) + s(b-\gamma) - b)\, y(k_t) - k_t + (a(1-s) - R_t(k_t))\, B_t$$

$$B_{t+1} - B_t = (R(k_t) - a - 1)B_t + by(k_t)$$

We will first analyze the existence and uniqueness of steady-state equilibrium and, then, we will analyze the stability of equilibrium and the dynamic behavior of capital and debt. The steady-state of capital stock and debt level in the economy is that bundle, k, b, where $k_{t+1} - k_t = 0$ and $B_{t+1} - B_t = 0$ simultaneously.

The locus where the change of debt is zero, $B_{t+1} - B_t = 0$ is given by

$$B = \frac{by(k)}{(1+a) - R(k)} \equiv \Gamma(k)$$

The properties of $\Gamma(k)$ are the following:

1. $\lim_{k\to 0} \Gamma(k) = 0$ and $\lim_{k\to\infty} \Gamma(k) = \infty$.

2. $\Gamma(k)$ is discontinuous at $k = \check{k}$ where $\check{k} : (1+a) - R(\check{k}) = 0$ Under the Cobb-douglas function production function $\check{k} = \left(\frac{(1+a)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-1-\gamma}}$

3. For $0 < k < \check{k}$ then $\Gamma(k) < 0$ and for $\check{k} < k < \infty$ then $\Gamma(k) > 0$.

Proof. Note that $y(k) > 0$ for any $k > 0$ and $\frac{\partial((1+a)-R(k_t))}{\partial k} = -\acute{R}(k_t) > 0$ (monotonic function). Also, $\lim_{k\to 0}(1+a) - R(k_t) = -\infty$ and $\lim_{k\to\infty}(1+a) - R(k_t) = (1+a) > 0$. This means that for $0 < k < \check{k}$ then $(1+a) - R(k_t) < 0$ and for $\check{k} < k < \infty$ then $(1+a) - R(k_t) > 0$

As $R(k) = \alpha\tilde{A}k^{\frac{\alpha}{1-\gamma}-1}$ we have $(1+a) - \alpha\tilde{A}k^{\frac{\alpha}{1-\gamma}-1} > 0 \Rightarrow \hat{k} < \left(\frac{(1+a)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-1-\gamma}}$

4. The limit behavior of $\Gamma(k)$ from the left and the right of discontinuity is given by: $\lim_{k\to\check{k}^-} \Gamma(k) = -\infty$ and $\lim_{k\to\check{k}^+} \Gamma(k) = \infty$.

5. The first order derivative $\Gamma(k)$ is given by:
$\frac{\partial\Gamma(k)}{\partial k} = b\frac{y'(k_t)((1+a) - R(k_t)) + (R'(k_t))y(k_t)}{((1+a) - R(k_t))^2}$ which after simplification (see footnote)[16]

$$\frac{\partial\Gamma(k)}{\partial k} = b\frac{\left(\frac{(1+a)}{(1-\gamma)} - \frac{R(k)}{\alpha}\right)R(k)}{((1+a) - R(k_t))^2}$$

[16]Simplification of the numerator of the first order derivative

$$\frac{\alpha}{(1-\gamma)}\frac{y(k)}{k}((1+a) - R(k)) + \alpha(\tfrac{\alpha}{1-\gamma} - 1)\frac{y(k)}{k^2}y(k) \Rightarrow$$
$$\left(\frac{1}{(1-\gamma)}((1+a) - R(k)) + (\tfrac{\alpha}{1-\gamma} - 1)\frac{y(k)}{k}\right)\frac{\alpha y(k)}{k}$$
$$\left((\tfrac{(1+a)}{(1-\gamma)} - \tfrac{R(k)}{(1-\gamma)}) + (\tfrac{\alpha}{1-\gamma} - 1)\tfrac{R(k)}{\alpha}\right)R(k)$$
$$\left(\tfrac{(1+a)}{(1-\gamma)} - \tfrac{R(k)}{(1-\gamma)} + \tfrac{\alpha}{1-\gamma}\tfrac{R(k)}{\alpha} - \tfrac{R(k)}{\alpha}\right)R(k)$$
$$\left(\tfrac{(1+a)}{(1-\gamma)} - \tfrac{R(k)}{(1-\gamma)} + \tfrac{R(k)}{1-\gamma} - \tfrac{R(k)}{\alpha}\right)R(k)$$
$$\left(\tfrac{(1+a)}{(1-\gamma)} - \tfrac{R(k)}{\alpha}\right)R(k)$$

$$\frac{\partial\Gamma(k)}{\partial k} = b\frac{\left(\frac{(1+a)}{(1-\gamma)} - \frac{R(k)}{\alpha}\right)R(k)}{((1+a) - R(k_t))^2}$$

For $0<k<\check{k}$ then $\frac{\partial\Gamma(k)}{\partial k}<0$.

This happens because $0<k<\check{k}$, $y^{'}(k_t)((1+a)-R(k_t))<0$ and $(R'(k_t))y(k_t)<0$ given that $y^{'}(k_t)>0$, $(1+a)-R(k_t)<0$ and $R'(k_t)<0$ and $y(k_t)>0$.

Definition 1 *Define* $k_{\min}\equiv\left(\frac{(1+a)}{(1-\gamma)\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}$

For $\check{k}<k<\infty$ then,
(i) $\frac{\partial\Gamma(k)}{\partial k}<0$ for $\check{k}<k<k_{\min}$
(ii) $\frac{\partial\Gamma(k)}{\partial k}>0$ for $k_{\min}<k<\infty$.

Proof. $\frac{\partial\Gamma(k)}{\partial k}<0$ if $y^{'}(k)((1+a)-R(k))+(R'(k))y(k)<0$ which following $R(k)=\alpha\tilde{A}k_t^{\frac{\alpha}{1-\gamma}-1}=\alpha\frac{y(k)}{k}$ and $R_k=\alpha(\frac{\alpha}{1-\gamma}-1)\tilde{A}k_t^{\frac{\alpha}{1-\gamma}-2}=\alpha(\frac{\alpha}{1-\gamma}-1)\frac{y(k)}{k^2}$, $y^{'}(k_t)=\dfrac{\alpha}{(1-\gamma)}\tilde{A}k_t^{\overline{(1-\gamma)}^{\alpha}-1}=\dfrac{\alpha}{(1-\gamma)}\frac{y(k)}{k}$ we have

$\dfrac{\alpha}{(1-\gamma)}\frac{y(k)}{k}((1+a)-R(k))+\alpha(\frac{\alpha}{1-\gamma}-1)\frac{y(k)}{k^2}y(k)<0\Rightarrow$
$\Rightarrow(1+a)\alpha-(1-\gamma)R(k)<0$ that is
$(1+a)\alpha-(1-\gamma)\alpha\tilde{A}k^{\frac{\alpha}{1-\gamma}-1}<0\Rightarrow k^{\frac{\alpha}{1-\gamma}-1}>\frac{(1+a)}{(1-\gamma)\tilde{A}}\Rightarrow k<\left(\frac{(1+a)}{(1-\gamma)\tilde{A}}\right)^{\frac{1}{\frac{\alpha}{1-\gamma}-1}}\Rightarrow k<\left(\frac{(1+a)}{(1-\gamma)\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}\equiv k_{\min}$. The opposite happens otherwise.

Second order derivative:
$\frac{\partial^2\Gamma(k)}{\partial k^2}=b\dfrac{\left(\frac{(1+a)\acute{R}}{(1-\gamma)}-\frac{2R\acute{R}}{\alpha}\right)(1+a-R)^2+\left(\frac{(1+a)}{(1-\gamma)}-\frac{R}{\alpha}\right)R2\left(1+a-R\right)\acute{R}}{(1+a-R)^4}$
taking common factor $\acute{R}$ and eliminating $(1+a-R)$

$$\frac{\partial^2\Gamma(k)}{\partial k^2}=\frac{b\acute{R}}{(1-\gamma)\alpha}\frac{R(\alpha-2(1-\gamma))+\alpha(1+a)}{(1+a-R)^3}$$

Analysis of $\frac{\partial^2\Gamma(k)}{\partial k^2}$. We analyze the case after the discontinuity, that is, for $\check{k}<k<\infty$
Then, for $\check{k}<k<\infty$ then $1+a-R>0$ then $\frac{\partial^2\Gamma(k)}{\partial k^2}>0$ if $R(\alpha-2(1-\gamma))+\alpha(1+a)<0$
$\Rightarrow R(2(1-\gamma)-\alpha)>\alpha(1+a)\Rightarrow R>\frac{\alpha(1+a)}{2(1-\gamma)-\alpha}\Rightarrow k<\left(\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}\equiv\tilde{k}$. So, for $\check{k}<k<\tilde{k}$, $\frac{\partial^2\Gamma(k)}{\partial k^2}>0$
We also, want to show that $\tilde{k}$ is indeed above the discontinuity $\check{k}$
First, we compare $\tilde{k}$ with $\check{k}$, we need $\tilde{k}>\check{k}\Rightarrow\left(\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}>\left(\frac{(1+a)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-1-\gamma}}\Rightarrow\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}<\frac{(1+a)}{\alpha\tilde{A}}\Rightarrow$

$\alpha<(2(1-\gamma)-\alpha)\Rightarrow 2\alpha<2(1-\gamma)\Rightarrow a-(1-\gamma)<0$ which holds.

Thus the function is convex for $\check{k} < k < \tilde{k}$, $\frac{\partial^2 \Gamma(k)}{\partial k^2} > 0$ and concave for $\tilde{k} < k < \infty$, $\frac{\partial^2 \Gamma(k)}{\partial k^2} < 0$. Last, $\lim_{k \to \infty} \frac{\partial^2 \Gamma(k)}{\partial k^2} = 0$.

Now we are going to analyze the locus where the change of capital stock is zero, $K_{t+1} - K_t = 0$, which is given by

$$\Theta(k) = \frac{(s(1-\alpha) - (1-s)b - s\gamma)\, y(k) - k}{(R(k) - a(1-s))}$$

where $y(k) = \tilde{A}k^{\overline{(1-\gamma)}^{\alpha}}$ and $y_k = \tilde{A}\frac{\alpha}{(1-\gamma)}k^{\overline{(1-\gamma)}^{\alpha}-1} = \frac{\alpha}{(1-\gamma)}\frac{y(k)}{k}$ and the limit behavior is: $\lim_{t\to 0} y(k) = 0, \lim_{t\to\infty} y(k) = \infty$ $\lim_{t\to 0} y_k = \infty$ and $\lim_{t\to\infty} y_k = 0$
and
$R(k) = \alpha\tilde{A}k_t^{\frac{\alpha}{1-\gamma}-1} = \alpha\frac{y(k)}{k}$ and $R_k = \alpha(\frac{\alpha}{1-\gamma}-1)\tilde{A}k_t^{\frac{\alpha}{1-\gamma}-2} = \alpha(\frac{\alpha}{1-\gamma}-1)\frac{y(k)}{k^2} = (\frac{\alpha}{1-\gamma}-1)\frac{R(k)}{k}$

1. $\lim_{k\to 0}\Theta(k) = \lim_{k\to 0}\Theta(k) = \frac{\Omega y(0)-0}{(R(0)-a(1-s))} = 0$ and $\lim_{k\to\infty}\Theta(k) = \frac{\frac{\partial((s(1-\alpha)-(1-s)b-s\gamma)y(k)-k)}{\partial k}}{\frac{\partial((R(k)-a(1-s)))}{\partial k}} = \lim_{k\to\infty}\frac{(s(1-\alpha)-(1-s)b-s\gamma)\acute{y}(k)-1}{\acute{R}(k)} = \lim_{k\to\infty}\frac{(s(1-\alpha)-(1-s)b-s\gamma)y''(k)}{R''(k)} = \infty$

2. $\Theta(k)$ is discontinuous at $k = \hat{k}$ where $\hat{k} : R(\hat{k}) - a(1-s) = 0$. Under a Cobb-douglas production function $\hat{k} = \left(\frac{a(1-s)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}$

Remark 1 *We show that the discontinuity of the debt locus is below the discontinuity of the k locus. That is* $\left(\frac{(1+a)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-1-\gamma}} < \left(\frac{a(1-s)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} \Rightarrow (1+a) > a(1-s) \Rightarrow (1+a) > a(1-s) \Rightarrow 1 > -as$ *where for* $a > 0$ *and* $\alpha \in (0,1)$ *this always holds.*

Assumption 1 *We assume a positive effect of income (investment) on the accumulation of capital stock which happens under the following condition* $(s(1-\alpha) + s(b-\gamma) - b > 0$ *or* $b < \frac{s(1-\alpha-\gamma)}{(1-s)} \equiv b^{\max}$

3. Define k_{AUT}: $(s(1-\alpha) - (1-s)b - s\gamma)\, y(k_{AUT}) - k_{AUT} = 0$ (in other words $B = 0$) which in the Cobb-douglas case is given by: $(s(1-\alpha) - (1-s)b - s\gamma)\,\tilde{A}k^{\overline{(1-\gamma)}^{\alpha-(1-\gamma)}} - 1 = 0 \Rightarrow k_{AUT} = \left(\frac{1}{\tilde{A}(s(1-\alpha)-(1-s)b-s\gamma)}\right)^{\frac{(1-\gamma)}{\alpha-(1-\gamma)}}$.

Assumption 2 *Parametric condition such that:* $\hat{k} > k_{AUT}$ *is:* $\left(\frac{a(1-s)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} > \left(\frac{1}{\tilde{A}(s(1-\alpha)-(1-s)b-s\gamma)}\right)^{\frac{(1-\gamma)}{\alpha-(1-\gamma)}}$

$\frac{a(1-s)}{\alpha\tilde{A}} < \frac{1}{\tilde{A}(s(1-\alpha)-(1-s)b-s\gamma)} \Rightarrow a(1-s)((s(1-\alpha)-(1-s)b-s\gamma)) < \alpha$ *which imposes limits on austerity* $a < \frac{\alpha}{(1-s)(s(1-\alpha)-(1-s)b-s\gamma)} \equiv a^{\max}$.

Then, because of concavity of $y(k)$ it is easy to show that the value of $\Theta(k)$ is given by the following remark.

Remark 2 *(i) for* $0 < k < k_{AUT}$ *then* $\Theta(k) > 0$ *and* $R(\hat{k}) - a(1-s) > 0$
(ii) for $k_{AUT} < k < \hat{k}$ *then* $\Theta(k) < 0$ *and* $R(\hat{k}) - a(1-s) > 0$
(iii) for $\hat{k} < k < \infty$ *then* $\Theta(k) > 0$ *and* $R(\hat{k}) - a(1-s) < 0$

4. The limit behavior of $\Theta(k)$ at the discontinuity is given by:

$\lim_{k \to \hat{k}^-} \Theta(k) = -\infty$ and $\lim_{k \to \hat{k}^+} \Theta(k) = \infty$.

5. The first order derivative of $\Theta(k)$:

Define $\Omega \equiv (s(1-\alpha) - (1-s)b - s\gamma)$

$\frac{\partial \Theta(k)}{\partial k} = \frac{(\Omega y_k - 1)(R(k) - a(1-s)) - (\Omega y(k) - k)R_k}{(R(k) - a(1-s))^2} =$

We then use the following equations
$R(k) = \alpha \frac{y(k)}{k}$, and $R_k = \alpha(\frac{\alpha}{1-\gamma} - 1)\tilde{A}k_t^{\frac{\alpha}{1-\gamma}-2} = \alpha(\frac{\alpha}{1-\gamma} - 1)\frac{y(k)}{k^2} = (\frac{\alpha}{1-\gamma} - 1)\frac{R(k)}{k}$, $y_k = \dfrac{\alpha}{(1-\gamma)} \frac{y(k)}{k} = \frac{1}{(1-\gamma)} R(k)$
Then, the derivative gets: (we express everything in $R(k)$)

$\frac{\partial \Theta(k)}{\partial k} = \frac{(\Omega \frac{1}{(1-\gamma)} R(k) - 1)(R(k) - a(1-s)) - (\Omega \frac{R(k)k}{\alpha} - k)(\frac{\alpha}{1-\gamma} - 1)\frac{R(k)}{k}}{(R(k) - a(1-s))^2} =$

$$\frac{\partial \Theta(k)}{\partial k} = \frac{\frac{\Omega}{\alpha} R^2 - (\frac{\Omega a(1-s) - \alpha}{(1-\gamma)} + 2)R + a(1-s)}{(R - a(1-s))^2}$$

Define $Z = \frac{\Omega}{\alpha}$, $C = a(1-s)$ and $\Xi = (\frac{\Omega C - \alpha}{(1-\gamma)} + 2) = (\frac{\alpha(ZC-1)}{(1-\gamma)} + 2)$

$$\frac{\partial \Theta(k)}{\partial k} = \frac{ZR^2 - \Xi R + C}{(R(k) - C)^2}$$

which is a quadratic equation with **at most two roots**.

5.1 (Limiting behavior) By applying the de hospital rule

$$\lim_{k \to 0} \frac{\partial \Theta(k)}{\partial k} = \frac{\Omega}{\alpha} > 0 \text{ and } \lim_{k \to \infty} \frac{\partial \Theta(k)}{\partial k} = \frac{1}{(a(1-s))} > 0$$

5.2 $\frac{\partial \Theta(k)}{\partial k} > 0$ if $ZR^2 - \Xi R + C > 0$ and $\frac{\partial \Theta(k)}{\partial k} < 0$ for $ZR^2 - \Xi R + C < 0$ which depends on the number of roots.
Discriminant: $\Xi^2 - 4ZC = (\frac{a(AC-1)}{(1-\gamma)} + 2)^2 - 4ZC = \frac{a^2(ZC^2 - 2ZC + 1)}{(1-\gamma)^2} + \frac{4a(ZC-1)}{(1-\gamma)} + 4 - 4ZC = \frac{a^2(ZC^2 - 2ZC + 1)}{(1-\gamma)^2} + \frac{4a(ZC-1)}{(1-\gamma)} + 4(1 - ZC)$

6.(Second order derivatives)

The first order derivative is given by:

$\frac{\partial \Theta(k)}{\partial k} = \frac{ZR^2 - \Xi R + C}{(R(k) - C)^2}$

Taking the second order derivative we obtain that: $\frac{\partial^2 \Theta(k)}{\partial k^2} = \frac{(Z2R\acute{R} - \Xi \acute{R})(R-C)^2 - (ZR^2 - \Xi R + C)(R(k)-C)2\acute{R}}{(R-C)^4}$ which, after some algebra is given by,

$$\frac{\partial^2 \Theta(k)}{\partial k^2} = \acute{R}\frac{R(\Xi - 2ZC) + C(\alpha \frac{ZC-1}{(1-\gamma)})}{(R-C)^3}$$

The derivative is negative until the discontinuity $0 < k < \hat{k}$ $(R - C > 0)$ of the k-zeros locus if: $R(\Xi - 2ZC) + C(\alpha(\frac{ZC-1}{(1-\gamma)}) > 0$ because $\acute{R} < 0$. Thus, we need that,

$$R > \frac{-C(\alpha(\frac{AC-1}{(1-\gamma)}))}{(\Xi - 2ZC)}$$

$$\alpha \tilde{A} k_t^{\frac{\alpha-(1-\gamma)}{1-\gamma}} > \frac{-C(\alpha(\frac{ZC-1}{(1-\gamma)}))}{(\Xi - 2ZC)}$$

$$k < \left(\frac{-C(\alpha(\frac{ZC-1}{(1-\gamma)}))}{\alpha\tilde{A}(\Xi - 2ZC)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} \equiv \tilde{k}$$

this is a necessary and sufficient condition for concavity. We now want to show if this is true for $0 < \tilde{k} < \hat{k}$ ($\tilde{k}$ below the discontinuity $\hat{k}$).

$\left(\frac{-C(\alpha(\frac{ZC-1}{(1-\gamma)}))}{(\Xi-2ZC)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} < \left(\frac{C}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}$

$(\alpha(\frac{AC-1}{(1-\gamma)})) < (\Xi - 2ZC)$

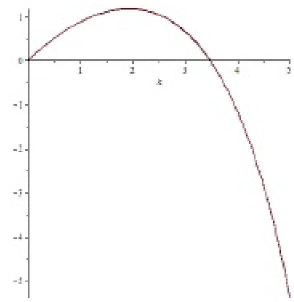
Note that $\Xi = (\frac{\Omega C - \alpha}{(1-\gamma)} + 2) = (\frac{\frac{\Omega}{\alpha}C - 1}{(1-\gamma)} + 2) = (\frac{\alpha(\frac{\Omega}{\alpha}C-1)}{(1-\gamma)} + 2) = (\frac{\alpha(ZC-1)}{(1-\gamma)} + 2)$

Substituting to the inequality $(\alpha(\frac{ZC-1}{(1-\gamma)})) < \frac{\alpha(ZC-1)}{(1-\gamma)} + 2 - 2ZC \Rightarrow 0 < +2 - 2ZC \Rightarrow 2ZC < 2 \Rightarrow ZC < 1$. Which holds from the assumption that limits austerity (see Remark 1) where we have $\frac{a(1-s)}{\alpha} < \frac{1}{(s(1-\alpha)-(1-s)b-s\gamma)} \Rightarrow \frac{C}{\alpha} < \frac{1}{\Omega} \Rightarrow \frac{\Omega}{\alpha} < \frac{1}{C} \Rightarrow Z < \frac{1}{C} \Rightarrow ZC < 1$.

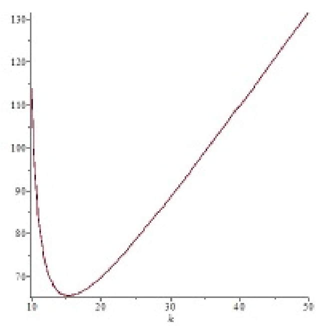
Lemma 1 *Under Remark 1, then* $\Theta(k)$ *is concave and inverse U-shaped for* $0 < k < \hat{k}$ *and convex (U-shaped) for* $\hat{k} < k < \infty$.

To illustrate this Lemma with diagram, the graph of the k-locus is given by:

Before the discontinuity



After the discontinuity



The steady states are defined by the following expression

$$F(k) = \Theta(k) - \Gamma(k)$$

$$F(k) = \frac{(\Omega)\, y(k) - k}{(R(k) - C)} - \frac{by(k)}{(1+a) - R(k)}$$

i. $F(0) = 0$

For $0 < k < \check{k}$, $\Theta(k) > 0$ and $\Gamma(k) < 0$ thus, $F(k) > 0$. Also, $\lim\limits_{k \to \check{k}^-} F(k) = +\infty$

Then, $\lim\limits_{k \to \check{k}^-} F(k) = -\infty$ and $\lim\limits_{k \to \check{k}^+} \acute{F}(k) > 0$. So, just after the discontinuity of the debt locus the $F(k)$ function is increasing.

Also, $\lim\limits_{k \to \hat{k}^-} F(k) = -\infty$, $\lim\limits_{k \to \hat{k}^-} \acute{F}(k) < 0$.

So, $F(k)$ is increasing from the discontinuity of the debt locus and it is decreasing at the discontinuity of the capital stock locus.

Since, $\check{k} < k < \hat{k}$ the derivative changes sign. We are going to explore if the maximum of the function is positive:

$$\acute{F}(k)=\frac{\partial\Theta(k)}{\partial k}-\frac{\partial\Gamma(k)}{\partial k},\frac{\partial\Theta(k_{\max})}{\partial k}-\frac{\partial\Gamma(k_{\max})}{\partial k}=0\Rightarrow\frac{ZR^2-\Xi R+C}{(R(k)-C)^2}-\left(\frac{(1+a)}{(1-\gamma)}-\frac{R}{\alpha}\right)R=0$$
$$ZR^2-\Xi R+C-\frac{R^3(1+a)}{(1-\gamma)\alpha}+\frac{2R^2C(1+a)}{(1-\gamma)\alpha}-\frac{RC^2(1+a)}{(1-\gamma)\alpha}+\frac{R^3(1-\gamma)R}{(1-\gamma)\alpha}-\frac{2R^2C(1-\gamma)R}{(1-\gamma)\alpha}+\frac{RC^2(1-\gamma)R}{(1-\gamma)\alpha}=0$$

$$F^{''}(k)=\acute{R}\frac{R(\Xi-2ZC)+C(\alpha\frac{ZC-1}{(1-\gamma)})}{(R-C)^3}-\frac{b\acute{R}}{(1-\gamma)\alpha}\frac{R(\alpha-2(1-\gamma))+\alpha(1+a)}{(1+a-R)^3}$$

.

Because we proved that $\acute{R}\frac{R(\Xi-2ZC)+C(\alpha\frac{ZC-1}{(1-\gamma)})}{(R-C)^3}<0$ after the discontinuity of the debt locus and between the k_{AUT}, then, for concavity of $F(k)$ we need $\frac{b\acute{R}}{(1-\gamma)\alpha}\frac{R(\alpha-2(1-\gamma))+\alpha(1+a)}{(1+a-R)^3}>0$ which from the analysis of the debt locus after the discontinuity hold for $R(\alpha-2(1-\gamma))+\alpha(1+a)<0\Rightarrow k<\left(\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}\equiv\tilde{k}$. Thus, for $k<\tilde{k}$ then $F^{''}(k)<0$. Thus, if that $\tilde{k}$ is below the discontinuity of the k-locus $\hat{k}=\left(\frac{a(1-s)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}$.

Thus, a sufficient parametric condition for concavity of $F(k)$ in the area between the discontinuities, $\check{k}<k<\hat{k}$, is that $\tilde{k}<\hat{k}$ that is

$\left(\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}<\left(\frac{a(1-s)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}\Rightarrow\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}>\frac{a(1-s)}{\alpha\tilde{A}}\Rightarrow(1+a)\alpha>a(1-s)(2(1-\gamma)-\alpha)$.

Lemma 2 *If $(1+a)\alpha>a(1-s)(2(1-\gamma)-\alpha)$ then in the area between the discontinuities $\check{k}<k<\hat{k}$, $F^{''}(k)<0$.*

This Lemma means that if an equilibrium exists will be multiple. Furthermore,the debt locus will be convex at the tangency and the k locus concave.

A sufficient parametric condition for concavity of $F(k)$ is to investigate between the discontinuities of debt locus and the k_{AUT} (because in the area between k_{AUT} and the discontinuity of k-locus the debt is negative and no equilibrium can exist). So, in this case a sufficient condition is $\tilde{k}<k_{AUT}$.

$\left(\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}<\left(\frac{1}{\tilde{A}(s(1-\alpha)-(1-s)b-s\gamma)}\right)^{\frac{(1-\gamma)}{\alpha-(1-\gamma)}}\Rightarrow\frac{(1+a)}{(2(1-\gamma)-\alpha)}>\frac{1}{(s(1-\alpha)-(1-s)b-s\gamma)}\Rightarrow$ $(1+a)\left(s(1-\alpha)-(1-s)b-s\gamma\right)>(2(1-\gamma)-\alpha)$.

If $(1+a)\left(s(1-\alpha)-(1-s)b-s\gamma\right)>(2(1-\gamma)-\alpha)$ then in the area between the $\check{k}<k<k_{AUT}$, $F^{''}(k)<0$.

Under Lemma 1 and Lemma 2 and conditions (i) and (ii) of Proposition 1 we show that the shape of the k locus is inversed U-Shaped and the shape of the b-locus decreasing after the area of the discontinuity. So two equilibrium steady-state exist. Figure 1 below graphically illustrates our theoretical result (note that before the discontinuity there cannot exist an equilibrium with $k>0$ because the debt-locus has negative values)

Appendix 3. Stability

In this section, we are going to analyze the stability properties and the type of each equilibrium. We are going to construct the phase diagram and analyze the arrows of motion.

The dynamic equation for debt is given by

$B_{t+1} - B_t = (R(k_t) - a - 1)B_t + by(k_t)$

Remind that, for $\check{k} < k < \infty$ then $(1 + a) - R(k) > 0$. Then, for $B_{t+1} - B_t > 0$, $R(k_t) - a - 1)B_t + by(k_t) > 0$ that is $B_t < \frac{by(k_t)}{(1+a)-R(k)}$. Thus, for any B_t lower then the $\Gamma(k)$ locus and because $\Gamma(k)$ is convex, the debt is decreasing (increasing under the $\Gamma(k)$ locus).

The dynamic equation for the capital stock is given by

$k_{t+1} - k_t = (s(1-\alpha) + s(b-\gamma) - b)\, y(k_t) - k_t + (a(1-s) - R_t(k_t))\, B_t$

For $k_{t+1}-k_t > 0$ if $(s(1-\alpha) + s(b-\gamma) - b)\, y(k_t) - k_t + (a(1-s) - R_t(k_t))\, B_t > 0$. Remind that, for $\check{k} < k < k_{AUT}$ then $\Theta(k) > 0$ and $R(\hat{k}) - a(1-s) > 0$. Dividing the inequality by $R(\hat{k}) - a(1-s) > 0$ we get $\frac{(s(1-\alpha)+s(b-\gamma)-b)y(k_t)-k_t}{R(\hat{k})-a(1-s)} - B_t > 0 \Rightarrow B_t < \frac{(s(1-\alpha)+s(b-\gamma)-b)y(k_t)-k_t}{R(\hat{k})-a(1-s)} \Rightarrow B_t < \frac{(s(1-\alpha)+s(b-\gamma)-b)y(k_t)-k_t}{R(\hat{k})-a(1-s)} \Rightarrow B_t < \Theta(k)$. Because $\Theta(k)$ is a concave function, for every B below the $\Theta(k)$ locus the capital stock is increasing and below the $\Theta(k)$ locus it is decreasing.

According to this analysis, the phase diagram and the arrows of motion are given by Figure 1. From, these analytical results illustrated in Figure 1 we deduct that there are two stable equilibria. The lower equilibrium is saddle-path stable and the second equilibrium is stable node (all arrows of motion direct to this equilibrium).

Appendix 4. Steady-State Effects of Policy Parameters

The equilibrium steady-state of capital is given by:

$$F(k) = \frac{(\Omega)\, y(k) - k}{(R(k) - C)} - \frac{by(k)}{(1+a) - R(k)}$$

where $\Omega(b, a) \equiv (s(1-\alpha) - (1-s)b - s\gamma) > 0$, $C(a) \equiv a(1-s) > 0$

Firstly, we examine the effect of parameter a on steady-state capital stock. From the implicit function theorem we have:

$\frac{\partial k}{\partial a} = -\frac{\frac{\partial F(k)}{\partial k}}{\frac{\partial F(k)}{\partial a}}$

$\frac{\partial F(k)}{\partial a} = \frac{\partial \frac{(\Omega)y(k)-k}{(R(k)-C)}}{\partial a} + \dfrac{by(k)}{((1+a) - R(k))^2} > 0$ because $\frac{\partial \frac{(\Omega)y(k)-k}{(R(k)-C)}}{\partial a} > 0$ at $0 < k < k_{AUT}$.

$\frac{\partial F(k)}{\partial k} > 0$ from $0 < k < k_{\max}$ and $\frac{\partial F(k)}{\partial k} < 0$ from $k_{\max} < k < k_{AUT}$.

Given that the one equilibrium, k_{ss}^{low} is below $k_{\max}$ and the other, k_{ss}^{high}, above $k_{\max}$ those two equilibria display different properties as for $k_{ss}^{low} < k_{\max}$ $\frac{\partial F(k)}{\partial k} > 0 \Rightarrow \frac{\partial k}{\partial a} < 0$. While, for $k_{ss}^{high} > k_{\max}$, $\frac{\partial F(k)}{\partial k} < 0 \Rightarrow \frac{\partial k}{\partial a} > 0$. This results to Proposition 3.

Secondly, we examine the effect of structural deficit parameter on steady-state capital stock. From the implicit function theorem we have:

$\frac{\partial k}{\partial b} = -\frac{\frac{\partial F(k)}{\partial k}}{\frac{\partial F(k)}{\partial b}}$

$\frac{\partial F(k)}{\partial b} = \frac{-(1-s)y(k)}{(R(k)-C)} - \dfrac{y(k)}{(1+a) - R(k)}$

We know that for $0 < k < k_{AUT}$, $R(\hat{k}) - C > 0$ and for $\check{k} < k < \infty$, $(1+a) - R(k) > 0$. Thus, in the area we are interested $\check{k} < k < k_{AUT}$ we have:

$\frac{\partial F(k)}{\partial b} < 0$, for $\check{k} < k < k_{AUT}$. As in Proposition 3, $\frac{\partial F(k)}{\partial k} > 0$ from $0 < k < k_{\max}$ and $\frac{\partial F(k)}{\partial k} < 0$ from $k_{\max} < k < k_{AUT}$. For $k_{ss}^{low} < k_{\max}$ $\frac{\partial F(k)}{\partial k} > 0$ and $\frac{\partial F(k)}{\partial b} < 0 \Rightarrow \frac{\partial k}{\partial b} > 0$. While, for $k_{ss}^{high} > k_{\max}$, $\frac{\partial F(k)}{\partial k} < 0$ and $\frac{\partial F(k)}{\partial b} < 0 \Rightarrow \frac{\partial k}{\partial b} < 0$. Thus as b increases the lower equilibrium k increases and the higher equilibrium k falls resulting to Proposition 4.

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Figure 1: Phase Diagram

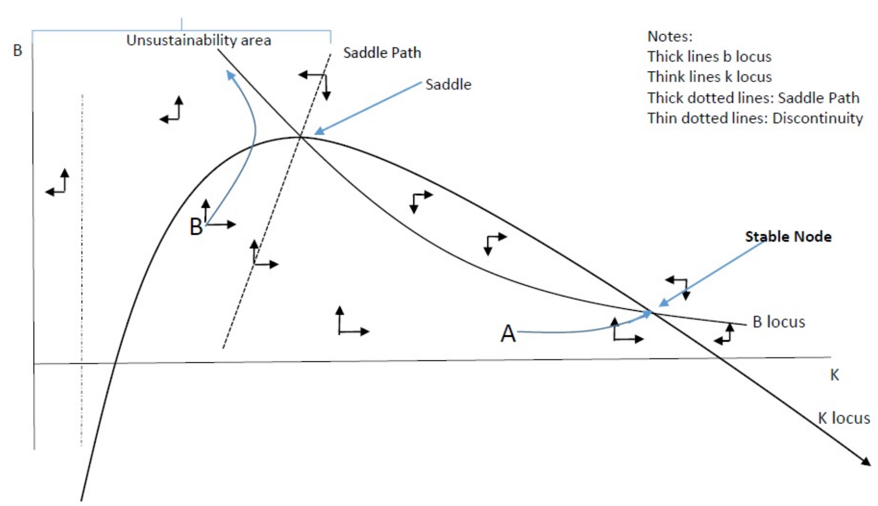

Figure 2: Increase in a

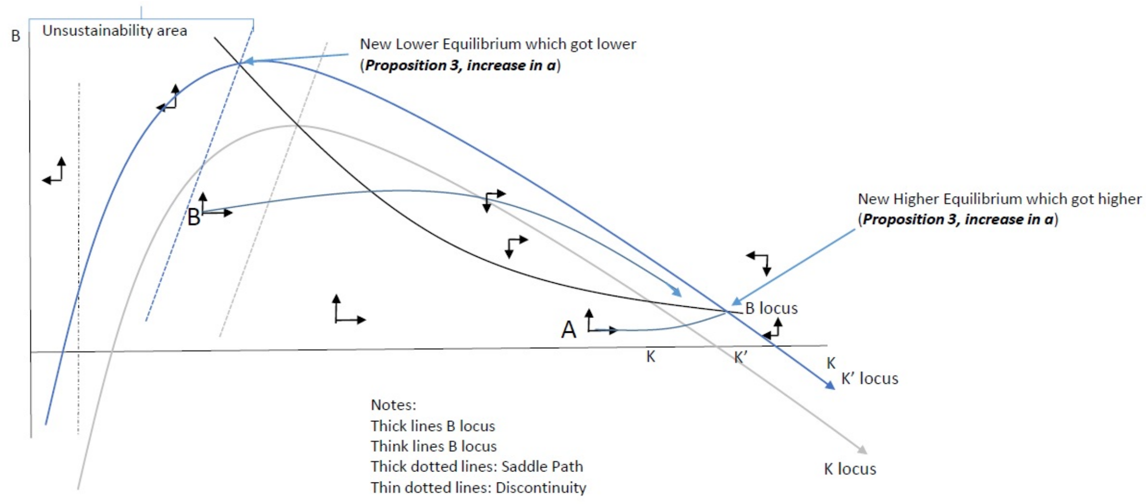

Figure 3.1

Country A: Dynamic adjustment towards the stable steady-state

with $a = 0.5$ and $b = 0.013$

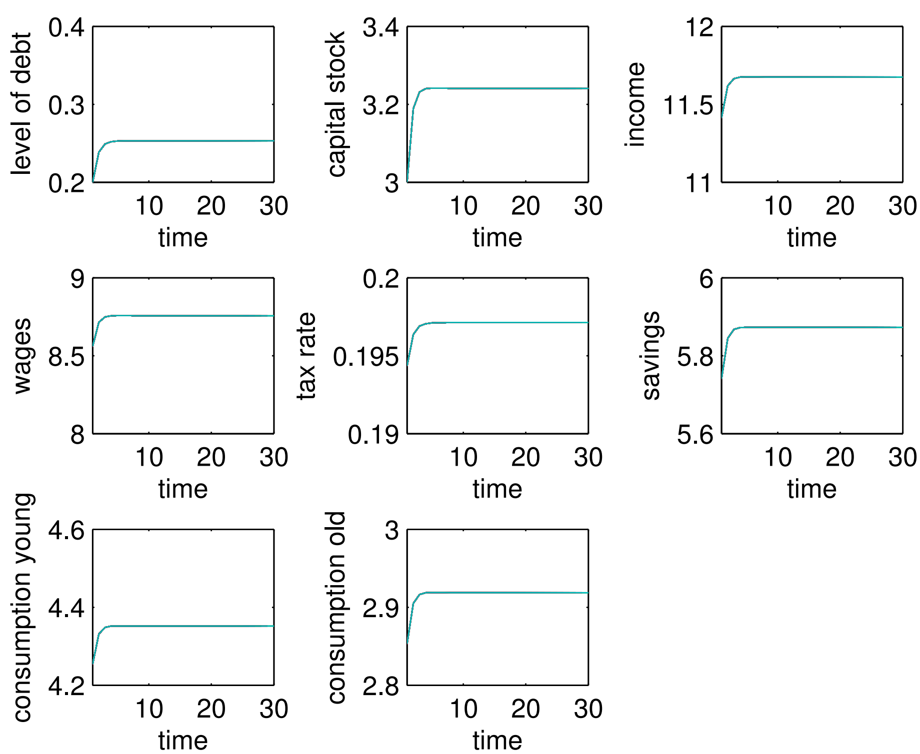

Figure 3.2

Country B: Dynamic adjustment towards exploding debt

with $a = 0.5$ and $b = 0.013$

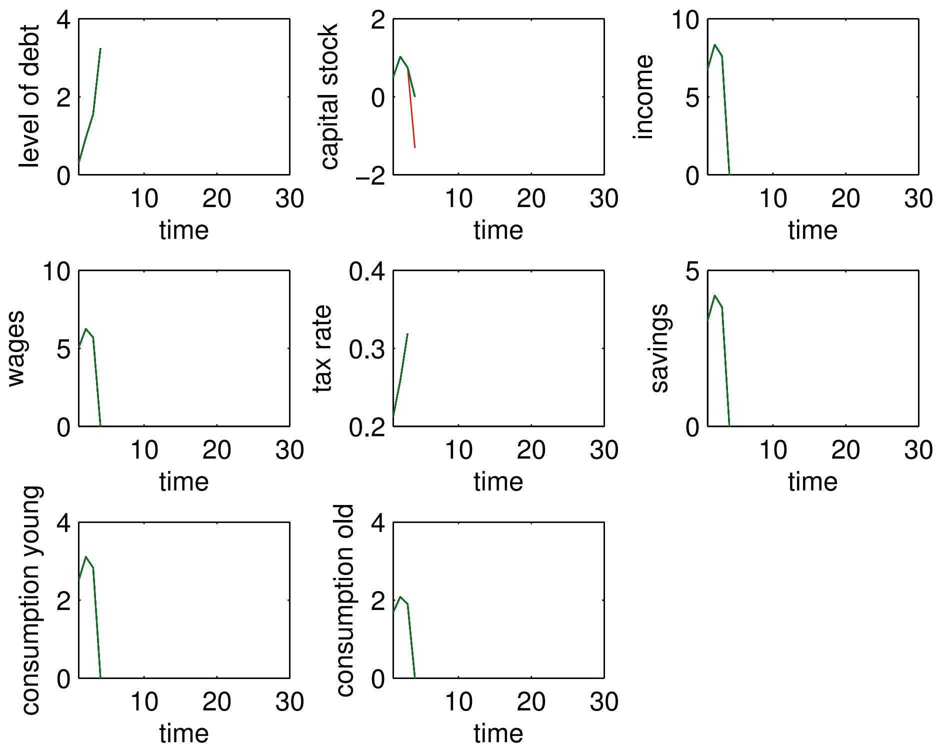

Figure 3.3

Country B: Dynamic adjustment towards the stable steady-state

with $a = 0.9$ and $b = 0.013$